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COVARIANT HARMONIC OSCILLATORS AND COUPLED HARMONIC OSCILLATORS

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Abstract

It is shown that the system of two coupled harmonic oscillators shares the basic symmetry properties with the covariant harmonic oscillator formalism which provides a concise description of the basic features of relativistic hadronic features observed in high-energy laboratories. It is shown also that the coupled oscillator system has the $SL(4,r)$ symmetry in classical mechanics, while the present formulation of quantum mechanics can accommodate only the $Sp(4,r)$ portion of the $SL(4,r)$ symmetry. The possible role of the $SL(4,r)$ symmetry in quantum mechanics is discussed.

1 Introduction

The covariant harmonic oscillator formalism developed by the present authors has been shown to be effective in explaining the basic phenomenological features of relativistic extended hadrons observed in high-energy laboratories. In particular, the formalism shows that the quark model and Feynman's parton picture are two different manifestations of one relativistic entity. In addition, the formalism constitutes a representation of Wigner's little group for a massive particle with internal space-time structure [1].

Since the classical mechanics of two coupled harmonic oscillators is discussed in Goldstein's text book [2], there is a tendency to believe that this oscillator problem is completely understood and that nothing new can be learned from it. We disagree. In this paper, we show that this coupled oscillator system can serve as an analog computer for the above-mentioned covariant oscillator formalism.

From the mathematical point of view, the standard approach is to construct a suitable representation of the symmetry group after writing down its generators. The first symmetry group in the present case is $Sp(4,r)$ with ten generators [3, 4, 5]. The second symmetry group is $SL(4,r)$ which contains a number of $Sp(4)$ -like subgroups. In constructing these groups, we shall note that each oscillator has its own $Sp(2)$ symmetry, and that the coupling of the two oscillator also has a

$Sp(2)$ -like symmetry. It was pointed out that these three $Sp(2)$ groups can be combined into one $Sp(4)$ group.

Since $Sp(4)$ is locally isomorphic to the deSitter group $O(3,2)$, it can explain the Lorentz transformation properties, particularly that of the covariant harmonic oscillator formalism. In this paper, we concentrate on the issue of a lack of information on one oscillator affecting the uncertainty and the entropy of the other oscillator.

2 Covariant Harmonic Oscillators

The covariant harmonic oscillator formalism has been discussed exhaustively in the literature, and it is not necessary to give another full-fledged treatment in the present paper. Instead, we shall concentrate on the issue of entropy in this paper. The entropy is a measure of our ignorance and is computed from the density matrix [6, 7]. The density matrix is needed when the experimental procedure does not analyze all relevant variables to the maximum extent consistent with quantum mechanics. The purpose of the present note is to discuss a concrete example of the entropy arising from our ignorance in relativistic quantum mechanics.

Let us consider a bound state of two particles. For convenience, we shall call the bound state the hadron, and call its constituents quarks. Then there is a Bohr-like radius measuring the space-like separation between the quarks. There is also a time-like separation between the quarks, and this variable becomes mixed with the longitudinal spatial separation as the hadron moves with a relativistic speed.

However, there are at present no quantum measurement theories to deal with the above-mentioned time-like separation. We shall study in the present paper how this ignorance is translated into the entropy. Within the framework of the covariant harmonic oscillator formalism [1], it will be shown that the entropy increases as the hadron gains its speed. The entropy defined in this way is a more fundamental quantity than the hadronic temperature [4]. It is independent of the question of whether the temperature can be defined [8].

Let us consider a hadron consisting of two quarks. If the space-time positions of two quarks are specified by x_a and x_b respectively, the system can be described by the variables [9]

$$X = (x_a + x_b)/2, \quad x = (x_a - x_b)/2\sqrt{2}. \quad (1)$$

The four-vector X specifies where the hadron is located in space and time, while the variable x measures the space-time separation between the quarks. In the convention of Feynman *et al* [9], the internal motion of the quarks bound by a harmonic oscillator potential of unit strength can be described by the Lorentz-invariant equation

$$\frac{1}{2} \left\{ x_\mu^2 - \frac{\partial^2}{\partial x_\mu^2} \right\} \psi(x) = \lambda \psi(x). \quad (2)$$

We use here the space-favored metric: $x^\mu = (x, y, z, t)$.

It is possible to construct a representation of the Poincaré group from the solutions of the above differential equation [1]. If the hadron is at rest, the solution should take the form

$$\psi(x, y, z, t) = \psi(x, y, z) \left(\frac{1}{\pi} \right)^{1/4} \exp(-t^2/2), \quad (3)$$

where $\psi(x, y, z)$ is the wave function for the three-dimensional oscillator with appropriate angular momentum quantum numbers. There are no excitations along the t direction. Indeed, the above wave function constitutes a representation of Wigner's $O(3)$ -like little group for a massive particle [1].

Since the three-dimensional oscillator differential equation is separable in both spherical and Cartesian coordinate systems, $\psi(x, y, z)$ consists of Hermite polynomials of x, y , and z . If the Lorentz boost is made along the z direction, the x and y coordinates are not affected, and can be dropped from the wave function. The wave function of interest can be written as

$$\psi^n(z, t) = \left(\frac{1}{\pi}\right)^{1/4} \exp(-t^2/2) \psi_n(z), \quad (4)$$

with

$$\psi^n(z) = \left(\frac{1}{\pi n! 2^n}\right)^{1/2} H_n(z) \exp(-z^2/2), \quad (5)$$

where $\psi^n(z)$ is for the n -th excited oscillator state. The full wave function $\psi^n(z, t)$ is

$$\psi_0^n(z, t) = \left(\frac{1}{\pi n! 2^n}\right)^{1/2} H_n(z) \exp\left\{-\frac{1}{2}(z^2 + t^2)\right\}. \quad (6)$$

The subscript 0 means that the wave function is for the hadron at rest. The above expression is not Lorentz-invariant, and its localization undergoes a Lorentz squeeze as the hadron moves along the z direction [1].

It is convenient to use the light-cone variables to describe Lorentz boosts. The light-cone coordinate variables are

$$u = (z + t)/\sqrt{2}, \quad v = (z - t)/\sqrt{2}. \quad (7)$$

In terms of these variables, the Lorentz boost along the z direction,

$$\begin{pmatrix} z' \\ t' \end{pmatrix} = \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}, \quad (8)$$

takes the simple form

$$u' = e^\eta u, \quad v' = e^{-\eta} v, \quad (9)$$

where η is the boost parameter and is $\tanh^{-1}(v/c)$. The wave function of Eq.(6) can be written as

$$\psi_0^n(z, t) = \left(\frac{1}{\pi n! 2^n}\right)^{1/2} H_n\left(\frac{(u + v)}{\sqrt{2}}\right) \exp\left\{-\frac{1}{2}(u^2 + v^2)\right\}. \quad (10)$$

If the system is boosted, the wave function becomes

$$\psi_\eta^n(z, t) = \left(\frac{1}{\pi n! 2^n}\right)^{1/2} H_n\left(\frac{(e^{-\eta} u + e^\eta v)}{\sqrt{2}}\right) \exp\left\{-\frac{1}{2}(e^{-2\eta} u^2 + e^{2\eta} v^2)\right\}. \quad (11)$$

As was discussed in the literature for several different purposes, this wave function can be expanded as [1]

$$\psi_\eta^n(z, t) = (1/\cosh \eta)^{n+1} \sum_k \left(\frac{(n+k)!}{n!k!}\right)^{1/2} (\tanh \eta)^k \psi_{n+k}(z) \psi_n(t). \quad (12)$$

In both Eqs. (10) and (11), the localization property of the wave function in the uv plane is determined by the Gaussian factor, and it is sufficient to study the ground state only for the essential feature of the boundary condition. Eq.(10) and Eq.(11) then respectively become

$$\psi_0^n(z, t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}(u^2 + v^2)\right\}. \quad (13)$$

If the system is boosted, the wave function becomes

$$\psi_\eta(z, t) = \left(\frac{1}{\pi}\right)^{1/2} \exp\left\{-\frac{1}{2}(e^{-2\eta}u^2 + e^{2\eta}v^2)\right\}. \quad (14)$$

We note here that the transition from Eq.(13) to Eq.(14) is a squeeze transformation. The wave function of Eq.(13) is distributed within a circular region in the uv plane, and thus in the zt plane. On the other hand, the wave function of Eq.(14) is distributed in an elliptic region. This ellipse is a "squeezed" circle with the same area as the circle. This Lorentz-squeezed wave function can be expanded as

$$\psi_\eta(z, t) = \frac{1}{\cosh \eta} \sum_k (\tanh \eta)^k \psi_k(z) \psi_k(t). \quad (15)$$

From this wave function, we can construct the pure-state density matrix

$$\rho_\eta(z, t; z', t') = \psi_\eta(z, t) \psi_\eta^*(z', t'), \quad (16)$$

which satisfies the condition $\rho^2 = \rho$:

$$\rho_\eta(z, t; z', t') = \int \rho_\eta(z, t; z'', t'') \rho_\eta(z'', t''; z', t') dz'' dt''. \quad (17)$$

However, there are at present no measurement theories which accommodate the time-separation variable t . Thus, we can take the trace of the ρ matrix with respect to the t variable. Then the resulting density matrix is

$$\begin{aligned} \rho_\eta(z, z') &= \int \psi_\eta(z, t) \{\psi_\eta(z', t)\}^* dt \\ &= \left(\frac{1}{\cosh \eta}\right)^2 \sum_k (\tanh \eta)^{2k} \psi_k(z) \psi_k^*(z'). \end{aligned} \quad (18)$$

The trace of this density matrix is one, but the trace of ρ^2 is less than one, as

$$\begin{aligned} Tr(\rho^2) &= \int \rho_\eta(z, z') \rho_\eta(z', z) dz' dz \\ &= \left(\frac{1}{\cosh \eta}\right)^4 \sum_k (\tanh \eta)^{4k}, \end{aligned} \quad (19)$$

which is less than one. This is due to the fact that we do not know how to deal with the time-like separation in the present formulation of quantum mechanics. Our knowledge is less than complete.

The standard way to measure this ignorance is to calculate the entropy defined as [6, 7]

$$S = -Tr(\rho \ln(\rho)).$$

If we pretend to know the distribution along the time-like direction and use the pure-state density matrix given in Eq.(16), then the entropy is zero. However, if we do not know how to deal with the distribution along t , then we should use the density matrix of Eq.(18) to calculate the entropy, and the result is

$$S \neq 2 \left\{ (\cosh \eta)^2 \ln(\cosh \eta) - (\sinh \eta)^2 \ln(\sinh \eta) \right\}. \quad (20)$$

In terms of the velocity v of the hadron,

$$S = -\ln[1 - (v/c)^2] - \frac{(v/c)^2 \ln(v/c)^2}{1 - (v/c)^2}. \quad (21)$$

We can also calculate the density matrix using the Gaussian form of the wave function given in Eq.(17), and the result is

$$\rho(z, z') = \left(\frac{1}{\pi \cosh 2\eta} \right)^{1/2} \exp \left\{ -\frac{1}{4} [(z + z')^2 / \cosh 2\eta + (z - z')^2 \cosh 2\eta] \right\}, \quad (22)$$

This expression also leads to the entropy given in Eq.(20)

The diagonal elements of the above density matrix is

$$\rho(z, z) = \left(\frac{1}{\pi \cosh 2\eta} \right)^{1/2} \exp \left(-z^2 / \cosh 2\eta \right). \quad (23)$$

The width of the distribution becomes $(\cosh \eta)^{1/2}$, and becomes wide-spread as the hadronic speed increases. Likewise, the momentum distribution becomes wide-spread. This simultaneous increase in the momentum and position distribution widths is called the parton phenomenon in high-energy physics. The position-momentum uncertainty becomes $\cosh \eta$. This increase in uncertainty is due to our ignorance about the physical but unmeasurable time-separation variable.

The use of an unmeasurable variable as a “shadow” coordinate is not new in physics and is of current interest [10, 11, 12, 13]. Feynman’s book on statistical mechanics contains the following paragraph [14].

When we solve a quantum-mechanical problem, what we really do is divide the universe into two parts - the system in which we are interested and the rest of the universe. We then usually act as if the system in which we are interested comprised the entire universe. To motivate the use of density matrices, let us see what happens when we include the part of the universe outside the system.

In the present paper, we have identified Feynman’s rest of the universe as the time-separation coordinate in a relativistic two-body problem. Our ignorance about this coordinate leads to a density matrix for a non-pure state, and consequently to an increase of entropy. It is interesting to note that the density matrix of Eq.(22) becomes that of the harmonic oscillator in a thermal equilibrium state if $(\tanh \eta)^2$ is identified as the Boltzmann factor [15].

We have thus far studied the properties of covariant harmonic oscillators where the longitudinal and time-like coordinates undergo squeeze transformations. The word “squeeze” is relatively new in physics. However, squeeze transformations are almost everywhere in physics. In the rest of this paper, we shall discuss the role of squeeze transformations in the system of two coupled harmonic oscillators. We shall see that the problem of covariant harmonic oscillators with two variables is the same as that of two coupled harmonic oscillators.

3 Linear Canonical and Non-Canonical Transformations in Classical Mechanics

For a dynamical system consisting of two pairs of canonical variables x_1, p_1 and x_2, p_2 , we can introduce the four-dimensional coordinate system:

$$(\eta_1, \eta_2, \eta_3, \eta_4) = (x_1, x_2, p_1, p_2). \quad (24)$$

Then the transformation of the variables from η_i to ξ_i is canonical if

$$MJ\tilde{M} = J, \quad (25)$$

where

$$M_{ij} = \frac{\partial}{\partial \eta_j} \xi_i,$$

and

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

For linear canonical transformations, we can work with the group of four-by-four real matrices satisfying the condition of Eq.(25). This group is called the four-dimensional symplectic group or $Sp(4)$. While there are many physical applications of this group, we are interested here in constructing the representations relevant to the study of two coupled harmonic oscillators.

It is more convenient to discuss this group in terms of its generators G , defined as

$$M = \exp(-i\alpha G), \quad (26)$$

where G represents a set of purely imaginary four-by-four matrices. The symplectic condition of Eq.(25) dictates that G be symmetric and anticommute with J or be antisymmetric and commute with J .

In terms of the Pauli spin matrices and the two-by-two identity matrix, we can construct the following four antisymmetric matrices which commute with J of Eq.(25).

$$\begin{aligned} J_1 &= \frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, & J_2 &= \frac{1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \\ J_3 &= \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, & J_0 &= \frac{i}{2} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \end{aligned} \quad (27)$$

The following six symmetric generators anticommute with J .

$$K_1 = \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \quad K_2 = \frac{i}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad K_3 = -\frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix},$$

and

$$Q_1 = \frac{i}{2} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad Q_2 = \frac{i}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad Q_3 = \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \quad (28)$$

These generators satisfy the commutation relations:

$$\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk}J_k, & [J_i, K_j] &= i\epsilon_{ijk}K_k, & [K_i, K_j] &= [Q_i, Q_j] = -i\epsilon_{ijk}J_k, \\
[J_i, J_0] &= 0, & [K_i, Q_j] &= i\delta_{ij}J_0, \\
[J_i, Q_j] &= i\epsilon_{ijk}Q_k, & [K_i, J_0] &= iQ_i, & [Q_i, J_0] &= -iK_i.
\end{aligned} \tag{29}$$

The group of homogeneous linear transformations with this closed set of generators is called the symplectic group $Sp(4)$. The J matrices are known to generate rotations while the K and Q matrices generate squeezes [4].

It is often more convenient to study the physics of four-dimensional phase space using the coordinate system

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (x_1, p_1, x_2, p_2). \tag{30}$$

The transformation from $(\eta_1, \eta_2, \eta_3, \eta_4)$ is

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix}, \tag{31}$$

and the J matrix becomes

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{32}$$

In this new coordinate system, the rotation generators take the form

$$\begin{aligned}
J_1 &= \frac{-1}{2} \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, & J_2 &= \frac{i}{2} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \\
J_3 &= \frac{-1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}, & J_0 &= \frac{-1}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}.
\end{aligned} \tag{33}$$

The squeeze generators become

$$\begin{aligned}
K_1 &= \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}, & K_2 &= \frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, & K_3 &= -\frac{i}{2} \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \\
Q_1 &= \frac{i}{2} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, & Q_2 &= \frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, & Q_3 &= \frac{i}{2} \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}.
\end{aligned} \tag{34}$$

In addition to the ten generators given in Eq.(33) and also in Eq.(34), we can consider the scale transformation in which both the position and momentum of the first coordinate are expanded and those of the second coordinate contracted. The Hamiltonian given in Eq.(46) suggests such a transformation, and the transformation can be generated by

$$S_0 = \frac{i}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{35}$$

This matrix generates scale transformations in phase space. The transformation leads to a radial expansion of the phase space of the first coordinate [16] and contracts the phase space of the second coordinate. What is the physical significance of this operation? As we discussed in Sec. 7, the expansion of phase space leads to an increase in uncertainty and entropy. Mathematically speaking, the contraction of the second coordinate should cause a decrease in uncertainty and entropy. Can this happen? The answer is clearly No, because it will violate the uncertainty principle. This question, will be addressed in future publications.

In the meantime, let us study what happens when the matrix S_0 is introduced into the set of matrices given in Eq.(33) and Eq.(34). It commutes with $J_0, J_3, K_1, K_2, Q_1,$ and Q_2 . However, its commutators with the rest of the matrices produce four more generators:

$$\begin{aligned} [S_0, J_1] &= \frac{i}{2} \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, & [S_0, J_2] &= \frac{1}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \\ [S_0, K_3] &= \frac{1}{2} \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, & [S_0, Q_3] &= \frac{1}{2} \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \end{aligned} \quad (36)$$

If we take into account the above five generators in addition to the ten generators of $Sp(4)$, there are fifteen generators. They form the closed set of commutation relations for the the group $SL(4, r)$. This $SL(4, r)$ symmetry of the coupled oscillator system may have interesting physical implications.

4 $SL(4, r)$ Formulation of Two Coupled Oscillators

Let us consider a system of two coupled harmonic oscillators. The Hamiltonian for this system is

$$H = \frac{1}{2} \left\{ \frac{1}{m_1} p_1^2 + \frac{1}{m_2} p_2^2 + A' x_1^2 + B' x_2^2 + C' x_1 x_2 \right\}. \quad (37)$$

where

$$A' > 0, \quad B' > 0, \quad 4A'B' - C'^2 > 0. \quad (38)$$

By making scale changes of x_1 and x_2 to $(m_1/m_2)^{1/4} x_1$ and $(m_2/m_1)^{1/4} x_2$ respectively, it is possible to make a canonical transformation of the above Hamiltonian to the form [17, 18]

$$H = \frac{1}{2m} \{p_1^2 + p_2^2\} + \frac{1}{2} \{Ax_1^2 + Bx_2^2 + Cx_1x_2\}, \quad (39)$$

with $m = (m_1 m_2)^{1/2}$. We can decouple this Hamiltonian by making the coordinate transformation:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (40)$$

Under this rotation, the kinetic energy portion of the Hamiltonian in Eq.(39) remains invariant. Thus we can achieve the decoupling by diagonalizing the potential energy. Indeed, the system becomes diagonal if the angle α becomes

$$\tan \alpha = \frac{C}{B - A}. \quad (41)$$

This diagonalization procedure is well known.

We now introduce the new parameters K and η defined as

$$K = \sqrt{AB - C^2/4}, \quad \exp(-2\eta) = \frac{A + B + \sqrt{(A - B)^2 + C^2}}{\sqrt{4AB - C^2}}, \quad (42)$$

in addition to the rotation angle α . In terms of this new set of variables, A , B and C take the form

$$\begin{aligned} A &= K \left(e^{2\eta} \cos^2 \frac{\alpha}{2} + e^{-2\eta} \sin^2 \frac{\alpha}{2} \right), \\ B &= K \left(e^{2\eta} \sin^2 \frac{\alpha}{2} + e^{-2\eta} \cos^2 \frac{\alpha}{2} \right), \\ A &= K (e^{-2\eta} - e^{2\eta}) \sin \alpha. \end{aligned} \quad (43)$$

the Hamiltonian can be written as

$$H = \frac{1}{2m} \{q_1^2 + q_2^2\} + \frac{K}{2} \{e^{2\eta} y_1^2 + e^{-2\eta} y_2^2\}, \quad (44)$$

where y_1 and y_2 are defined in Eq.(40), and

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & -\sin(\alpha/2) \\ \sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \quad (45)$$

This form will be our starting point. The above rotation together with that of Eq.(40) is generated by J_0 .

If we measure the coordinate variable in units of $(mK)^{1/4}$, and use $(mK)^{-1/4}$ for the momentum variables, the Hamiltonian takes the form

$$H = \frac{\omega}{2} e^\eta (e^{-\eta} q_1^2 + e^\eta y_1^2) + \frac{\omega}{2} e^{-\eta} (e^\eta q_2^2 + e^{-\eta} y_2^2), \quad (46)$$

where $\omega = \sqrt{K/m}$. If $\eta = 0$, the system becomes decoupled, and the Hamiltonian becomes

$$H = \frac{\omega}{2} (p_1^2 + x_1^2) + \frac{\omega}{2} (p_2^2 + x_2^2). \quad (47)$$

In Sec. 8, we will be dealing with the problem of what happens when no observations are made on the second coordinate. If the system is decoupled, as the above Hamiltonian indicates, the physics in the first coordinate is solely dictated by the Hamiltonian

$$H_1 = \frac{\omega}{2} (p_1^2 + x_1^2). \quad (48)$$

It is important to note that the Hamiltonian of Eq.(47) cannot be obtained from Eq.(46) by canonical transformation. For this reason, the Hamiltonian of the form

$$H' = \frac{\omega}{2} (e^{-\eta} q_1^2 + e^\eta y_1^2) + \frac{\omega}{2} (e^\eta q_2^2 + e^{-\eta} y_2^2) \quad (49)$$

may play a useful role in our discussion. This Hamiltonian can be transformed into the decoupled form of Eq.(47) through a canonical transformation.

5 Quantum Mechanics of Coupled Oscillators

It is remarkable that both the Hamiltonian H of Eq.(46) and H' of Eq.(49) lead to the same Schrödinger wave function. If y_1 and y_2 are measured in units of $(mK)^{1/4}$, the ground-state wave function for this oscillator system is

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2} (e^\eta y_1^2 + e^{-\eta} y_2^2) \right\}. \quad (50)$$

The wave function is separable in the y_1 and y_2 variables. However, for the variables x_1 and x_2 , the story is quite different. If we write this wave function in terms of x_1 and x_2 , then

$$\psi(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2} \left[e^\eta \left(x_1 \cos \frac{\alpha}{2} - x_2 \sin \frac{\alpha}{2} \right)^2 + e^{-\eta} \left(x_1 \sin \frac{\alpha}{2} + x_2 \cos \frac{\alpha}{2} \right)^2 \right] \right\}. \quad (51)$$

If $\eta = 0$, this wave function becomes

$$\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2} (x_1^2 + x_2^2) \right\}. \quad (52)$$

For other values of η , the wave function of Eq.(51) can be obtained from the above expression by a unitary transformation.

$$\sum_{m_1 m_2} A_{m_1 m_2}(\alpha, \eta) \psi_{m_1}(x_1) \psi_{m_2}(x_2), \quad (53)$$

where $\psi_m(x)$ is the m^{th} excited state wave function. The coefficients $A_{m_1 m_2}(\eta)$ satisfy the unitarity condition

$$\sum_{m_1 m_2} |A_{m_1 m_2}(\alpha, \eta)|^2 = 1. \quad (54)$$

It is possible to carry out a similar expansion in the case of excited states [1].

As for unitary transformations applicable to wave functions, let us go back the generators of canonical transformations in classical mechanics. As was stated before, they are also applicable to the Wigner phase-space distribution function. The canonical transformation of the Wigner function is translated into a unitary transformation of the Schrödinger wave function. There are therefore ten generators of unitary transformations applicable to Schrödinger wave functions [4, 3].

The Wigner phase-space picture is often more convenient for studying the problems of coupled harmonic oscillators. Unitary transformations in the Schrödinger picture can be achieved through canonical transformations in phase space. It has been known that canonical transformations are uncertainty-preserving transformations. They are also entropy-preserving transformations [5]. Are there then non-canonical transformations in quantum mechanics?

In the present case of coupled harmonic oscillators, we assume that we are not able to measure the x_2 coordinate. It is often more convenient to use the Wigner phase-space distribution function to study the density matrix, especially when we want to study the uncertainty products in detail [18, 14].

For two coordinate variables, the Wigner function is defined as [18]

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \int \exp\{-2i(p_1 y_1 + p_2 y_2)\} \\ \times \psi^*(x_1 + y_1, x_2 + y_2)\psi(x_1 - y_1, x_2 - y_2)dy_1 dy_2. \quad (55)$$

The Wigner function corresponding to the oscillator wave function of Eq.(51) is

$$W(x_1, x_2; p_1, p_2) = \left(\frac{1}{\pi}\right)^2 \exp\left\{-e^\eta(x_1 \cos \frac{\alpha}{2} - x_2 \sin \frac{\alpha}{2})^2 \right. \\ \left. - e^{-\eta}(x_1 \sin \frac{\alpha}{2} + x_2 \cos \frac{\alpha}{2})^2 - e^{-\eta}(p_1 \cos \frac{\alpha}{2} - p_2 \sin \frac{\alpha}{2})^2 \right. \\ \left. - e^\eta(p_1 \sin \frac{\alpha}{2} + p_2 \cos \frac{\alpha}{2})^2\right\}. \quad (56)$$

If we do not make observations in the $x_2 p_2$ coordinates, the Wigner function becomes

$$W(x_1, p_1) = \int W(x_1, x_2; p_1, p_2)dx_2 dp_2. \quad (57)$$

The evaluation of the integral leads to

$$W(x_1, p_1) = \left\{ \frac{1}{\pi^2(1 + \sinh^2 \eta \sin^2 \alpha)} \right\}^{1/2} \\ \times \exp\left\{-\left(\frac{x_1^2}{\cosh \eta - \sin \eta \cos \alpha} + \frac{p_1^2}{\cosh \eta + \sin \eta \cos \alpha}\right)\right\}. \quad (58)$$

This Wigner function gives an elliptic distribution in the phase space of x_1 and p_1 . This distribution gives the uncertainty product of

$$(\Delta x)^2(\Delta p)^2 = \frac{1}{4}(1 + \sinh^2 \eta \sin^2 \alpha). \quad (59)$$

This expression becomes 1/4 if the oscillator system becomes uncoupled with $\alpha = 0$. Because x_1 is coupled with x_2 , our ignorance about the x_2 coordinate, which in this case acts as Feynman's rest of the universe, increases the uncertainty in the x_1 world which, in Feynman's words, is the system in which we are interested.

In the Wigner phase-space picture, the uncertainty is measured in terms of the area in phase space where the Wigner function is sufficiently different from zero. According to the Wigner function for a thermally excited oscillator state, the temperature and entropy are also determined by the degree of the spread of the Wigner function phase space.

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